GESELLSCHAFT für
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Coupling rate-independent and rate-dependent processes: Existence results

Aim

Existence results for general class of doubly nonlinear equations

$$-\varrho\ddot{u}(t) \in \partial_u\mathcal{E}(t,u(t),z(t)) + \partial_{\dot{u}}\mathcal{V}(\dot{u}),$$
$$0 \in \partial_z\mathcal{E}(t,u(t),z(t)) + \partial_{\dot{z}}\mathcal{R}(\dot{z}(t)),$$

rate-independent evolution of $z \,\widehat{=}\, \mathcal{R}$ linear growth,
viscous evolution of $u \,\widehat{=}\, \mathcal{V}$ superlinear growth + inertia.
Notion of solution combines ideas from gradient flows [1] & rate-independent systems [2] and generalizes [3].

References:

[1] A. Mielke, R. Rossi, G. Savaré, Calc. Var. PDE 46(1-2):253–310, 2013.
[2] A. Mielke, Handbook of Differential Equations, Evolutionary Equations, 2:461–559, 2005.
[3] T. Roubíček, M²AS, 32:825-862, 2009.
[4] R. Rossi, M. Thomas, work in progress.

(Weak) energetic solutions [4]

Banach $X \Subset Z$, (V refl.), $U \subset V \subset W$, $U \Subset W$ Hilbert

Definition: $q=(u,z):[0,T]\to V\times Z$ is an **energetic solution** to the rate-independent system with viscosity & inertia $(V,W,Z,\mathcal{E},\mathcal{V},\mathcal{R})$ & initial data, if

$$(\mathrm{R})\begin{cases} z \in L^\infty(0,T;X)\cap \mathrm{BV}(0,T;Z) \\ \&\ u \in L^\infty(0,T;U)\cap W^{1,1}(0,T;V) \\ \&\ \ddot{u} \in L^\infty(0,T;W)\end{cases}$$

and if $q=(u,z)$ satisfies:

1. weak momentum balance for a.a. $t\in(0,T)$:
$$\rho\ddot{u}(t) + \mathrm{D}_{\dot{u}}\mathcal{V}(\dot{u}(t)) + \partial_u\mathcal{E}(t,u(t),z(t)) \ni 0 \quad \text{in } V^*$$
2. semistability for all $t\in[0,T]$:
$$\forall\tilde{z}\in Z: \qquad \mathcal{E}(t,u(t),z(t)) \le \mathcal{E}(t,u(t),\tilde{z}) + \mathcal{R}(\tilde{z}-z(t))$$
3. upper energy dissipation estimate for all $t\in[0,T]$:
$$\tfrac{\rho}{2}\|\dot{u}(t)\|_W^2 + \mathcal{E}(t,q(t)) + \int_0^t \mathcal{V}(\dot{u}(s)) + \mathcal{V}^*(-\xi(s)-\rho\ddot{u})\,\mathrm{d}s + \mathrm{Diss}_{\mathcal{R}}(z;[0,T])$$
$$\le \tfrac{\rho}{2}\|\dot{u}(0)\|_W^2 + \mathcal{E}(0,q(0)) + \int_0^t \partial_s\mathcal{E}(s,q(s))\,\mathrm{d}s$$

with $\xi(s)\in\partial_u\mathcal{E}(s,u(s),z(s))$ for a.e. $s\in(0,T)$ fulfilling 1. and $\mathrm{Diss}_{\mathcal{R}}(z;[0,T]) := \sup_{\text{part. of }[0,t]}\mathcal{R}(z(s_i)-z(s_{i-1}))$.

$q=(u,z)$ of regularity (R) is a **weak energetic solution** to the rate-independent system with viscosity & inertia $(V,W,Z,\mathcal{E},\mathcal{V},\mathcal{R})$ & initial data, if only 2. & 3. with $\xi(s)\in\partial_u\mathcal{E}(s,u(s),z(s))$ for a.e. $s\in(0,T)$ hold true.

Theorem (Existence results [4]):

- Let the assumtions **($\mathbf{A}_{\mathcal{R}}$)**, **($\mathbf{A}_{\mathcal{V}}$)**, **($\mathbf{A}_{\mathcal{E}}$)** & **(C1-3)** hold true. Then $(V,W,Z,\mathcal{E},\mathcal{V},\mathcal{R})$ has a **weak energetic solution**.
- In addition, let the following chain rule inequality hold true:

$$\int_0^t \langle \mathrm{D}_u\mathcal{E}(s,u(s),z(s)) + \rho\ddot{u}, \dot{u}(s)\rangle_V$$
$$\le \tfrac{\rho}{2}\|\dot{u}(t)\|_W^2 + \mathcal{E}(t,u(t),z(t)) - \tfrac{\rho}{2}\|\dot{u}(0)\|_W^2 - \mathcal{E}(0,u(0),z(0))$$
$$- \int_0^t \partial_s\mathcal{E}(s,u(s),z(s))\,\mathrm{d}s + \mathrm{Diss}_{\mathcal{R}}(z,[0,t]).$$

Then any weak energetic solution (u,z) also satisfies 1. and 3. is valid as an identity. In particular, (u,z) is an **energetic solution**.

Assumptions & Proof

($\mathbf{A}_{\mathcal{R}}$) $\mathcal{R}$ is pos. 1-homog. & convex on Z.

($\mathbf{A}_{\mathcal{V}}$) $\mathcal{V}:V\to[0,\infty)$ convex, superlinear growth,
$\mathcal{V}^*:V^*\to[0,\infty)$, $\mathcal{V}^*(\xi_v) := \sup_{v\in V}(\langle\xi_v,v\rangle_V - \mathcal{V}(v))$,
$\exists\, C_V, C_V^*>0\ \forall(v,\xi)\in V\times V^*$:
$$\mathcal{V}(v)+\mathcal{V}^*(\xi) \ge C_V|\langle\xi,v\rangle_V| - C_V^*,$$

($\mathbf{A}_{\mathcal{E}}$) (CES) Sublevels of $\mathcal{E}(t,\cdot)$ are compact in $U\times X$,
(UCP) Uniform control of the power: $\exists\, c_0\in\mathbb{R},\ c_1>0$
$\forall(t_q,q)\in[0,T]\times V\times Z$ with $\mathcal{E}(t_q,q)<\infty$:
$$\forall\, t\in[0,T]:\ |\partial_t\mathcal{E}(t,q)| \le c_1(c_0+\mathcal{E}(t,q)).$$
(EUFS) Enhanced uniform Fréchet-subdifferentiability:
$\exists\,\Lambda>0$, $\forall\, t\in[0,T]$, $\forall(u,z),(v,z)$ with finite energy $\forall\,\xi\in\partial_u\mathcal{E}(t,u,z)$:
$$\mathcal{E}(t,v,z)-\mathcal{E}(t,u,z) \ge \langle\xi,v-u\rangle_V - \Lambda\|v-u\|_W\mathcal{V}(v-u)^{1/2}$$
(SE) Subgradient estimate: $\exists\, C_1,C_2,C_3>0, \sigma\in[1,\infty)$,
$\forall\, t\in[0,T]$, $\forall(u,z)$ w. finite energy, $\forall\,\xi\in\partial_u\mathcal{E}(t,u,z)$:
$$\|\xi\|_{V^*}^\sigma \le C_1\mathcal{E}(t,u,z) + C_2\|u\|_V + C_3.$$

Compatibility conditions:

(C1) Convergence of the power:
$$\tilde{q}_n \rightharpoonup \tilde{q} \text{ in } V\times Z \ \Rightarrow\ \partial_t\mathcal{E}(t,\tilde{q}_n)\to\partial_t\mathcal{E}(t,\tilde{q})$$

(C2) Closedness of semistable sets:
If $(u_n,z_n)_n$ are semistable, $(u_n,z_n)\rightharpoonup(u,z)$ in $V\times Z$, then (u,z) is semistable for $(V\times Z,\mathcal{E},\mathcal{R})$.

(C3) Weak$_V$-Weak$_{V^*}$-closedness of $\partial_u\mathcal{E}$:
$\forall(t_n,u_n,z_n)_n$ with unif. bbd. $\mathcal{E}$ in $[0,T]$, $\forall\,\xi_n\in V^*$:
$$\begin{Bmatrix} t_n\to t,\ u_n\rightharpoonup u \text{ in } V,\ z_n\rightharpoonup z \text{ in } Z \\ \xi_n\in\partial_u\mathcal{E}(t_n,u_n,z_n),\ \xi_n\rightharpoonup\xi \text{ in } V^*\end{Bmatrix} \Rightarrow \xi\in\partial_u\mathcal{E}(t,u,z).$$

Strategy of the proof:

- Time-discrete scheme: For $n=1,\dots,N_\tau$ find (u_τ^n,z_τ^n) s.t.
$$z_\tau^n \in \operatorname*{argmin}_{z\in Z}\left(\tau\mathcal{R}\left(\tfrac{z-z_\tau^{n-1}}{\tau}\right)+\mathcal{E}(t_n,u_\tau^{n-1},z)\right), \tag{D1}$$
$$0 \in \rho\tfrac{u_\tau^n-2u_\tau^{n-1}+u_\tau^{n-2}}{\tau} + \partial\mathcal{V}\big(\tfrac{u_\tau^n-u_\tau^{n-1}}{\tau}\big) + \partial_u\mathcal{E}(t_{,n},u_\tau^n,z_\tau^n). \tag{D2}$$
- Deduce time-discrete version of 2. & 3.:
(D1) + triangle inequality $\Rightarrow$ discr. semistability
+ (D2), use (EUFS) $\Rightarrow$ discr. upper energy dissipation est.:

$$\tfrac{\rho}{2}\|u_\tau'(t)\|_W^2 + \int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)}\mathcal{V}(u_\tau'(r)) + \mathcal{V}^*(-\bar{\xi}_\tau(r)-\rho v_\tau'(r))\,\mathrm{d}r$$
$$+ \int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)}\mathcal{R}(z_\tau'(r))\,\mathrm{d}r + \mathcal{E}(\bar{t}_\tau(t),\bar{u}_\tau(t),\bar{z}_\tau(t))$$
$$\le \tfrac{\rho}{2}\|u_\tau'(s)\|_W^2 + \mathcal{E}(\bar{t}_\tau(s),\bar{u}_\tau(s),\bar{z}_\tau(s)) + \int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)}\partial_t\mathcal{E}(r,\underline{u}_\tau(r),\underline{z}_\tau(r))\,\mathrm{d}r$$
$$+ \tfrac{1}{2}\int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)}\mathcal{V}(u_\tau'(r))\,\mathrm{d}r + C\tau\int_{\bar{t}_\tau(s)}^{\bar{t}_\tau(t)}\|u_\tau'\|_W^2\,\mathrm{d}r.$$

- A priori estimates via Gronwall by (UCP), (CES), (SE) & **($\mathbf{A}_{\mathcal{V}}$)**.
- Limit passage in discr. version of 2. & 3. by **(C1-3)**.

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