GESELLSCHAFT für ANGEWANDTE MATHEMATIK und MECHANIK e.V.
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A system theoretic approach to H^∞-calculus

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Motivation

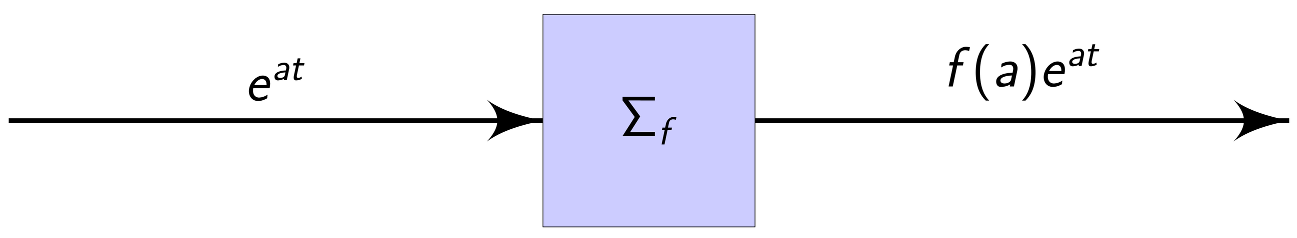
Alternative method to define

$$f(A)$$

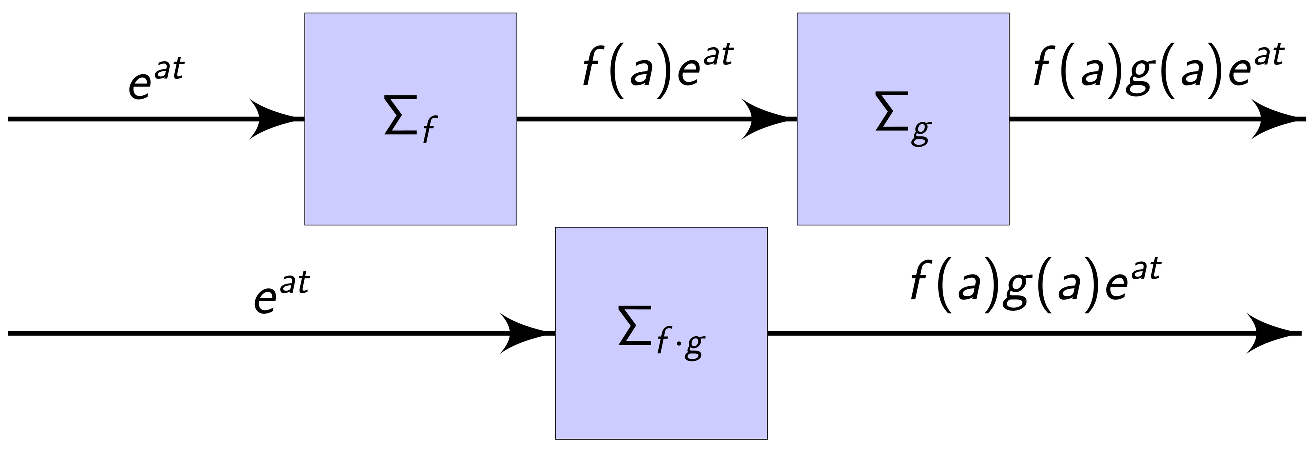
for $f : \mathbb{C}_- \to \mathbb{C}$ bounded, analytic and A being a linear operator, e.g. $A = \Delta$ on $L^2(\mathbb{R})$.

Appl.: numerical analysis, maximal regularity of PDEs.

The idea



- Σ_f is linear system with transfer function $f \in H^\infty$.
- Exponential input yields exponential output, $a < 0$.



- $\rightsquigarrow$ multiplicativity: $(f \cdot g)(a) \mapsto f(a) \cdot g(a)$.

Idea: Replace e^{at} by e^{At} to define $f(A)$

- $A \in \mathcal{G}_{exp}$, i.e. is the generator of an exponentially stable C_0-semigroup e^{At} on a Banach space X.
- If $f = \mathfrak{L}(h)$ for some bounded measure h, then

$$\Sigma_f : \qquad y(t) = (h * u)(t),$$

where u denotes the input and y the output of the system.

$$\rightsquigarrow f(A)e^{At} = \int_0^\infty h(t-s)e^{As}\, ds.$$

$\rightsquigarrow$ extension of the *Hille-Phillips*-calculus.

Is $t \mapsto f(A)e^{At}$ continuous?

$\Leftrightarrow$ $f(A)$ is bounded operator, i.e.

$$D(A) = X \quad \text{and} \quad \|f(A)x\| \leq const \cdot \|x\| \quad \forall x \in X.$$

- No in general (∞-dimensions): **Unbounded calculus**.

[1] Haase, Rozendaal, *Functional calculus for semigroup generators via transference*, *J. Funct. Anal.*, vol. 265, no. 12, pp. 3345–3368, 2013.

[2] Schwenninger, Zwart, *Weakly admissible $\mathcal{H}^-_\infty$-calculus on reflexive Banach spaces*, *Indag. Math.*, vol. 23, no. 4, pp. 796–815, 2012.

[3] Schwenninger, Zwart, *On measuring the unboundedness of the H^∞-caclulus*, 2014, in preparation.

[4] Zwart, *Toeplitz operators and $\mathcal{H}^\infty$-calculus*, *J. Funct. Anal.*, vol. 263, no. 1, pp. 167–182, 2012.

Admissible calculus

Output $y(t) = f(A)e^{At}$ is L^2 in the following sense.

Theorem 1 *Let $A \in \mathcal{G}_{exp}$, $f \in H^\infty$. Then, $f(A)$ is bounded from $D(A)$ to X and for all $x \in D(A)$ and $y \in X'$,*

$$\left(\int_0^\infty |\langle f(A)e^{At}x, y\rangle|^2 dt\right)^{\frac{1}{2}} \leq const \cdot \|x\| \cdot \|y\| \cdot \|f\|_\infty$$

*That means, $f(A)$ is **weakly admissible**.*

Theorem 2 *If C is bounded from $D(A)$ to Y and weakly admissible, then $Cf(A)$ is weakly admissible with*

$$\|Cf(A)\|_{w-adm} \leq \|C\|_{w-adm} \cdot \|f\|_\infty,$$

where $\|C\|_{w-adm}$ is the smallest constant such that

$$\|\langle Ce^{\cdot A}x, y\rangle\|_{L^2} \leq const \cdot \|x\| \cdot \|y\|, \quad x \in X, y \in Y'.$$

Hilbert spaces: *weakly admissible* replaced by *admissible*, [4].

Boundedness of calculus

Definition 3 *For C bounded from $D(A)$ to Y, (C, A) is called **exactly observable by direction** if*
$\exists k, K > 0\ \forall x \in D(A)\ \exists y_x \in Y', \|y_x\| = 1,$

$$k\|x\| \leq \|\langle Ce^{\cdot A}x, y_x\rangle\|_{L^2(0,\infty)} \leq K\|x\|.$$

Theorem 4 *Let (C, A) be exactly observable by direction. Then, $f \mapsto f(A)$ is a **bounded** H^∞**-calculus** with*

$$\|f(A)\| \leq \frac{K}{k}\|f\|_\infty.$$

Hilbert space: use *exact observability*.

Theorem 5 *If $A \in \mathcal{G}_{exp}$ and e^{tA} **analytic**, then for $f \in H^\infty$*

$$\|f(A)e^{tA}\| \leq \kappa(t) \cdot \|f\|_\infty,$$

with $\kappa(t) = \mathcal{O}(|\log(t)|)$ as $t \to 0^+$. See also [3] and [1].

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- Education
  - Diploma in mathematics, TU Wien, 2011
  - current: PhD at University Twente (NL)
- Research interests
  - C_0-semigroups, functional calculus
  - Infinite dimensional systems theory

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