GESELLSCHAFT für
ANGEWANDTE MATHEMATIK und MECHANIK e.V.
INTERNATIONAL ASSOCIATION of APPLIED MATHEMATICS and MECHANICS

# Relative Entropy in Compressible Multi-Phase Flows

## Motivation

We study compressible liquid-vapor flows described by diffuse interface models. Such a description is computationally advantageous as only one system of PDEs needs to be solved on the whole computational domain. Its solution contains the position of the phase boundary. A disadvantage of this approach is that the associated energy functional is non-convex creating difficulties in analysis and numerics.

Classical stability results for compressible flows are based on the relative entropy [1], requiring convexity of the energy. Higher order 'capillarity' effects in the model at hand allow us to compensate for the non-convex energy and to derive relative entropy based stability results.

## Mathematical Setting

We consider a one dimensional model problem

$$\begin{aligned} \tau_t - v_x &= 0 \\ v_t - W'(\tau)_x &= \mu v_{xx} - \gamma \tau_{xxx} \end{aligned} \qquad (1)$$

where $\tau$ is the specific volume, $v$ is the velocity, $\mu \geq 0$ is the viscosity parameter, $\gamma > 0$ is the capillarity parameter, and $W$ is the energy density given by a (non-convex) constitutive relation. We consider $(1)$ on the flat torus, denoted by $\mathbb{T}$.

Solutions of $(1)$ satisfy the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\mathbb{T}} W(\tau) + \frac{1}{2}v^2 + \frac{\gamma}{2}(\tau_x)^2 \,\mathrm{d}\,x = -\int_{\mathbb{T}} \mu (v_x)^2 \,\mathrm{d}\,x \leq 0. \qquad (2)$$

## Relative Entropy

The relative entropy between two solutions $(\tau, v)$ and $(\overline{\tau}, \overline{v})$ of $(1)$ is defined as

$$\eta\left(\begin{pmatrix}\tau\\ v\end{pmatrix}, \begin{pmatrix}\overline{\tau}\\ \overline{v}\end{pmatrix}\right) := \int_{\mathbb{T}} W(\tau) - W(\overline{\tau}) - W'(\overline{\tau})(\tau - \overline{\tau}) + \frac{1}{2}(v - \overline{v})^2 + \frac{\gamma}{2}(\tau_x - \overline{\tau}_x)^2 \,\mathrm{d}\,x.$$

The rate of change of relative entropy between any two strong solutions can be controlled. As $W$ is non-convex this is insufficient for controlling the difference between the solutions. Removing the $W$ terms from the relative entropy we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\mathbb{T}} \frac{1}{2}(v - \overline{v})^2 + \frac{\gamma}{2}(\tau_x - \overline{\tau}_x)^2 \,\mathrm{d}\,x \leq \int_{\mathbb{T}} (v - \overline{v})(W'(\tau) - W'(\overline{\tau}))_x \,\mathrm{d}\,x. \qquad (3)$$

Combining Gronwall's and Poincaré's inequalities we obtain:

**Lemma:** Let $(\tau, v)$, $(\overline{\tau}, \overline{v})$ be solutions of $(1)$ with initial data $\tau_0, \overline{\tau}_0 \in H^3(\mathbb{T})$ and $v_0, \overline{v}_0 \in H^2(\mathbb{T})$ with $\tau_0$ and $\overline{\tau}_0$ having the same mean value. Then, there exists some $C > 0$ such that

$$\|v(t,\cdot) - \overline{v}(t,\cdot)\|_{L^2} + \|\tau(t,\cdot) - \overline{\tau}_0(t,\cdot)\|_{H^1} \leq \left(\|v_0 - \overline{v}_0\|_{L^2} + \|\tau_0 - \overline{\tau}_0\|_{H^1}\right)\exp(Ct/\gamma). \qquad (4)$$

## Non-Local Model

We consider a family of models in which capillarity is modeled by non-local terms. Its solutions are parametrised in $\varepsilon > 0$ :

$$\begin{aligned} \tau^\varepsilon_t - v^\varepsilon_x &= 0 \\ v^\varepsilon_t - (W'(\tau^\varepsilon))_x &= \mu v^\varepsilon_{xx} - (\phi_\varepsilon * \tau^\varepsilon - \tau^\varepsilon)_x, \end{aligned} \qquad (5)$$

where $\phi$ is a symmetric, non-negative mollification kernel with compact support satisfying

$$\phi_\varepsilon(\cdot) := \frac{1}{\varepsilon}\phi\left(\frac{\cdot}{\varepsilon}\right), \quad \int_{\infty}^{\infty} \phi(s)\,\mathrm{d}\,s = 1, \quad \int_{\infty}^{\infty} \phi(s)s^2\,\mathrm{d}\,s = 2\gamma.$$

Solutions of $(5)$ satisfy the energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{\mathbb{T}} W(\tau^\varepsilon) + \frac{1}{2}(v^\varepsilon)^2 + \frac{\gamma}{2}(\phi_\varepsilon * \tau^\varepsilon - \tau^\varepsilon)^2 \,\mathrm{d}\,x = -\int_{\mathbb{T}} \mu (v^\varepsilon_x)^2 \,\mathrm{d}\,x \leq 0.$$

## Model Convergence

Using semi-group theory we can prove that, provided the initial data are sufficiently regular, $(1)$ and $(5)$ have strong solutions for arbitrarily long times. Using a modified relative entropy analogous to $(3)$ we can prove:

**Theorem:** Let $T, \mu, \gamma > 0$ be given. Let $(\tau, v)$ and $(\tau^\varepsilon, v^\varepsilon)$ be solutions to $(1)$ and $(5)$, respectively, with identical initial data in $H^3(\mathbb{T}) \times H^2(\mathbb{T})$. Then, the following estimate holds uniformly for all $t \in (0, T)$ :

$$\|\tau^\varepsilon(t,\cdot) - \tau(t,\cdot)\|_{L^2} + \|v^\varepsilon(t,\cdot) - v(t,\cdot)\|_{L^2} = \mathcal{O}(\varepsilon^{1/4}),$$

i.e., solutions of $(5)$ converge to solutions of $(1)$ for $\varepsilon \to 0$.

## References

[1] C. M. Dafermos. The second law of thermodynamics and stability. *Arch. Rational Mech. Anal.*, 70(2):167–179, 1979.

[2] J. Giesselmann. A relative entropy approach to convergence of a low order approximation to a nonlinear elasticity model with viscosity and capillarity. *SIAM J. Math. Anal.*, 46(5):3518–3539, 2014.

[3] J. Giesselmann. Relative entropy in multiphase models of 1d elastodynamics: Convergence of a local to a nonlocal model. *J. Differential Equations*, 258:3589–3606, 2015.

**Jan Giesselmann**

- Education
  - PhD in Mathematics, U Stuttgart
  - PostDoc at Univ of Crete and WIAS Berlin
  - Research and Teaching Associate U Stuttgart
- Research Interests
  - Compressible Multi-Phase Flows
  - Hyperbolic Conservation Laws