GESELLSCHAFT für
ANGEWANDTE MATHEMATIK und MECHANIK e.V.
INTERNATIONAL ASSOCIATION of APPLIED MATHEMATICS and MECHANICS

Variational Integrators

for Simulation and Control

Motivation

"Approximate the action instead of the equations of motion" A. J. Lew

Variational Integrators (VIs) offer many advantages

- preserve momenta, if the system has symmetries,
- backward error analysis reveals discrete time paths as exact solutions of a nearby Hamiltonian,
- energy error is bounded, thus excellent longtime behavior.

Moreover, VIs are eminently suited for optimal control, since they

- result in one-step maps that can be directly utilized as discrete state-space models,
- yield an unified computational framework for simulation of the mechanical system and optimization of the control.

Higher Order VIs

Construction in two steps

1. approximation of the state variables in time

$$\mathbf{q}(t) \approx \mathbf{q}^d(t) = \sum_{n=0}^{p} M_n(t)\mathbf{q}_{k+n/p}$$

2. time-step-wise quadrature of the action-integral..

$$\begin{aligned}\Delta S &= \int_{t_k}^{t_{k+1}} L\big(\mathbf{q}(t), \dot{\mathbf{q}}(t), t\big)\,\mathrm{d}t \\ &\approx \int_{t_k}^{t_{k+1}} L\big(\mathbf{q}^d(t), \dot{\mathbf{q}}^d(t), t\big)\,\mathrm{d}t \\ &\approx \sum_{m=1}^{g} w_m L\big(\mathbf{q}^d(t_m), \dot{\mathbf{q}}^d(t_m), t_m\big) = L_d\end{aligned}$$

..and the virtual work of the nonconservative forces

$$\begin{aligned}\delta W^{\mathrm{nc}} &= \int_{t_k}^{t_{k+1}} \mathbf{F}\cdot\delta\mathbf{q}\,\mathrm{d}t \approx \int_{t_k}^{t_{k+1}} \mathbf{F}\cdot\delta\mathbf{q}^d\,\mathrm{d}t \\ &\approx \sum_{m=1}^{g} w_m \mathbf{F}(t_m)\cdot\delta\mathbf{q}^d(t_m) = \sum_{n=0}^{p} \mathbf{F}^d_{k+n/p}\delta\mathbf{q}^d_{k+n/p}.\end{aligned}$$

The stationarity condition of the action functional yields Discrete Euler-Lagrange-Equations (position-momentum form)

$$\begin{aligned}\mathbf{p}_k &= -D_1 L_d(\mathbf{q}_k, \mathbf{q}_{k+1/p}\dots, \mathbf{q}_{k+1}) - \mathbf{F}^d_k \\ \mathbf{0} &= D_2 L_d(\mathbf{q}_k, \mathbf{q}_{k+1/p}\dots, \mathbf{q}_{k+1}) + \mathbf{F}^d_{k+1/p} \\ &\dots \\ \mathbf{0} &= D_p L_d(\mathbf{q}_k, \mathbf{q}_{k+1/p}\dots, \mathbf{q}_{k+1}) + \mathbf{F}^d_{k+\frac{p-1}{p}} \\ \mathbf{p}_{k+1} &= D_{p+1} L_d(\mathbf{q}_k, \mathbf{q}_{k+1/p}\dots, \mathbf{q}_{k+1}) + \mathbf{F}^d_{k+1}\end{aligned}$$

where $D_1 L_d = \frac{\partial L_d}{\partial \mathbf{q}_k}$, $D_2 L_d = \frac{\partial L_d}{\partial \mathbf{q}_{k+1/p}}, \dots, D_{p+1} L_d = \frac{\partial L_d}{\partial \mathbf{q}_{k+1}}$.

References: J. Marsden, T. Murphey, S. Leyendecker

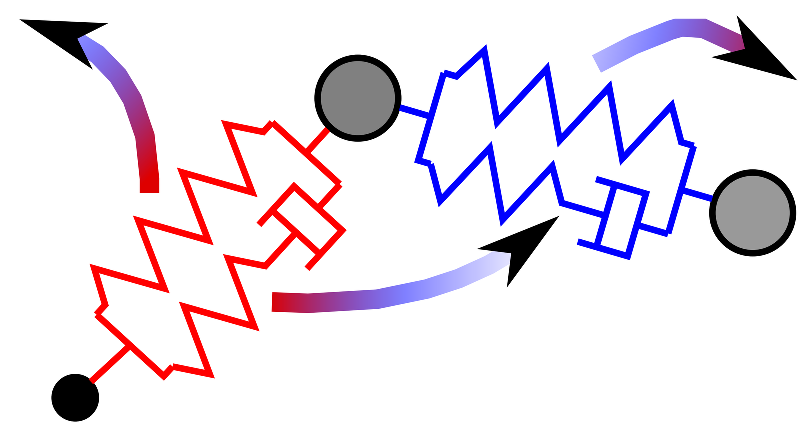
Thermo-viscoelastic Systems

Variational formulation

$$\delta\int_{t_0}^{t_1}(T^* - \psi)\,\mathrm{d}t + \int_{t_0}^{t_1}\delta W^{\mathrm{nc}}\,\mathrm{d}t = 0$$

where may be inserted the exemplaric expressions

kinetic coenergy	$T^* = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2)$
elastic strain energy	$\psi_e = \frac{K}{2l_0^2}(l - l_0)^2$
thermoelastic coupling	$\psi_{te} = -\beta(\vartheta - \vartheta_r)\frac{l - l_0}{l_0}$
heat capacity	$\psi_t = -\frac{k}{2\vartheta_r}(\vartheta - \vartheta_r)^2$
heat flux/source	$\delta W^{\mathrm{nc}}_t = \dot{s}\,\delta\alpha$
internal dissipation	$\delta W^{\mathrm{nc}}_v = F_v\,\delta v$



thermo-viscoelastic double pendulum as example

References: I. Romero, G. Maugin, A. Bertram

Outlook

Future work is related with

constraints

$$\delta S_c = \delta\int_{t_0}^{t_1} L(q, \dot{q}, t) - \sum_{i=1}^{N_c}\lambda_i\varphi_i\,\mathrm{d}t + \int_{t_0}^{t_1}\delta W^{\mathrm{nc}}\,\mathrm{d}t = 0$$

electromechanical systems

$$L = T^* + W_m^* - V - W_e$$

optimal control

cost $J = \int_{t_0}^{t_1}\ell(t, z, u)\,\mathrm{d}t$ dynamics $\dot{z} = f(z, u)$

References: T. Murphey, D. Liberzon, S. Ober-Blöbaum

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- Education
  - Diploma in Mechanical Engineering, TU Freiberg
  - Ph.D. from Karlsruhe Institute of Technology
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- Research Interests
  - Variational Integrators
  - Dynamics and Control