GESELLSCHAFT für
ANGEWANDTE MATHEMATIK und MECHANIK e.V.
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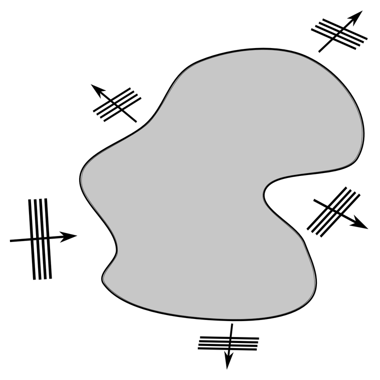
Regularization and Numerical Solution of the Inverse Scattering Problem Using Shearlet Frames

joint work with Gitta Kutyniok and Volker Mehrmann (both TU Berlin)

Introduction

Inverse Scattering

- The *inverse scattering problem* deals with reconstruction of objects from emitted waves.
- The waves can be, for instance, *acoustic* or *electromagnetic waves*.



- This problem is known to be an *ill -posed problem*. Therefore a regularization is necessary.

Ansatz

- Choose *cartoon-like images* as a model for scatterering objects.
- *Shearlet systems* allow for *sparse representations* of a large class of cartoon-like images.
- Employ the sparse representations of the scattering objects to *regularize* the problem.

Acoustic Scattering

Helmholtz equation in inhomogenious media with wave number k_0:

$$\Delta u + k_0^2(1-f)u = 0, \text{ with } u = u^s + u^{inc}$$

- *Contrast function* $f \in L^2(\mathbb{R}^2)$ describes scattering object.
- *Incident wave* u^{inc} is used to stimulate the medium.

The setup

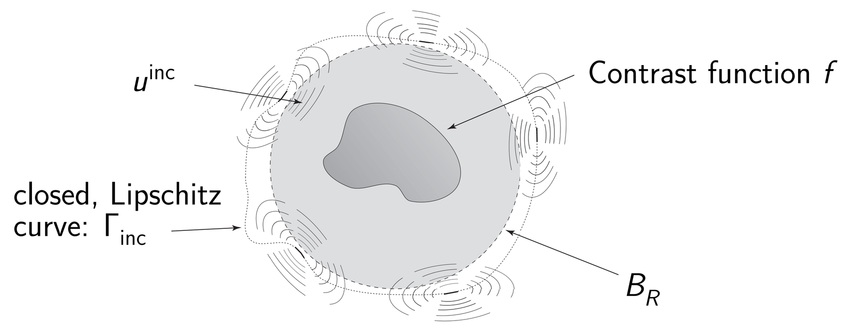


Figure 2: Setup with domain B_R contrast function f and single layer potentials as incident waves.

Solution operator $\mathscr{S}$:

$$\mathscr{S} : L^2(B_R) \times L^2(B_R) \to H^2_{\text{loc}}(B_R), \quad \mathscr{S}(f, u^{\text{inc}}) = u.$$

Measurement process and inverse problem

Let $\mathrm{HS}(L^2(\Gamma_{\text{inc}}), L^2(\Gamma_{\text{meas}}))$ denote the space of *Hilbert-Schmidt Operators* from $L^2(\Gamma_{\text{inc}})$ to $L^2(\Gamma_{\text{meas}})$.

Definition 1 (Lechleiter, Kazimierski, Kazimehmedovic; 2013)**.** *For a Lipschitz curve Γ_{meas} the multi-static measurement operator $\mathscr{N}$ is defined as*

$$\mathscr{N} : L^2_{\operatorname{Im}\geq 0}(B_R) \to \mathrm{HS}(L^2(\Gamma_{\text{inc}}), L^2(\Gamma_{\text{meas}})), \quad f \mapsto N_f,$$

where

$$N_f : L^2(\Gamma_{\text{inc}}) \to L^2(\Gamma_{\text{meas}}), \quad \phi \mapsto \mathscr{S}(f, SL_{\Gamma_{\text{inc}}}\phi)_{|\Gamma_{\text{meas}}}.$$

The *inverse scattering problem* is that of reconstructing f from N_f.

Model Assumption

We assume, that the scatterer is a *cartoon-like function*.

Definition 2. *A function $f \in L^2((0,1)^2)$ such that there exists $B \subset (0,1)^2$ such that ∂B twice differentiable, $f_1, f_2 \in C^2_c((0,1)^2)$ and*

$$f = f_1 + \chi_B f_2$$

is called cartoon-like function.

Cartoon-like images and representation systems

Error of best N-term approximation: For $f \in L^2(\mathbb{R}^2)$ and a dictionary $(\phi_i)_{i\in I}$ the best N-term approximation error is

$$\min_{\substack{\Lambda\subset\mathbb{N}, |\Lambda|=N,\\ \tilde{f}_N = \sum_{i\in\Lambda} c_i\phi_i}} \|f - \tilde{f}_N\|^2_{L^2(\mathbb{R}^2)}.$$

Theorem 1 (Donoho; 2001)**.** *For an arbitrary representation system $(\phi_i)_{i\in I} \subset L^2(\mathbb{R}^2)$, the minimally achievable asymptotic error of best N-term approximation for the class of cartoon-like functions is $O(N^{-2})$.*

Shearlets

Shearlet Systems are multiscale representation systems build upon *parabolic scaling* and *shearing*.

- Elements of the shearlet system:

$$\psi_{j,k,m} := 2^{\frac{3}{4}j}\psi(S_kA_j \cdot - cm), j \geq 0, k \in \mathbb{Z}, m \in \mathbb{Z}^2.$$

- *Parabolic scaling* and *shearing matrices*:

$$A_j = \begin{pmatrix} 2^j & 0 \\ 0 & 2^{\frac{j}{2}} \end{pmatrix}, S_k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}.$$

Theorem 2 (Kutyniok, Lim; 2011)**.** *There are shearlet systems such that the error of the best $N-$term approximation for any cartoon-like image decays as $O(N^{-2}\log(N)^3)$ as $N \to \infty$.*

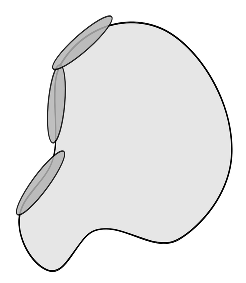


Figure 4: Shearlet elements overlapping the discontinuity curve of a cartoon-like function.

Schrödinger Scattering

- *Schrödinger equation:* $f \in L^2(\mathbb{R}^2)$ and a wave number $k > 0$, $d \in \mathbb{S}^1$, we search for $u = u^s + e^{ik\langle x,d\rangle}$

$$\Delta u + (f + k^2)u = 0, \quad \lim_{r\to\infty} r^{\frac{1}{2}}\left(\frac{\partial u^s(x)}{\partial r} - iku^s(x)\right) = 0.$$

- *Backscattering data:* Let $\theta \in [0, 2\pi]$, $\tau_\theta = (\cos(\theta), \sin(\theta))$, the backscattering amplitude is defined as

$$A(k,\theta) = \int_{\mathbb{R}^2} e^{ik\langle\tau_\theta, y\rangle} f(y)u(y)\,dy.$$

- *Born Approximation*

$$f_B(x) = \int_{\mathbb{R}^2} e^{-i\langle\xi,x\rangle} A\left(\frac{|\xi|}{2}, \theta\right) d\xi, \quad \xi = |\xi|\tau_\theta.$$

Theorem 3 (Kutyniok, Mehrmann, Petersen; 2015)**.** *Let $\varepsilon > 0$, let $s \in \mathbb{N}, s \geq 2$, and let, for some $x_0 \in \mathbb{R}^2$, $f \in L^2(\mathbb{R}^2) \cap H^{s+\varepsilon}(x_0)$ be compactly supported and real valued, then $f_B \in H^s(x_0)$.*

Corollary 1 (Kutyniok, Mehrmann, Petersen; 2015)**.** *If f is a cartoon-like function, then f_B is almost cartoon-like.*

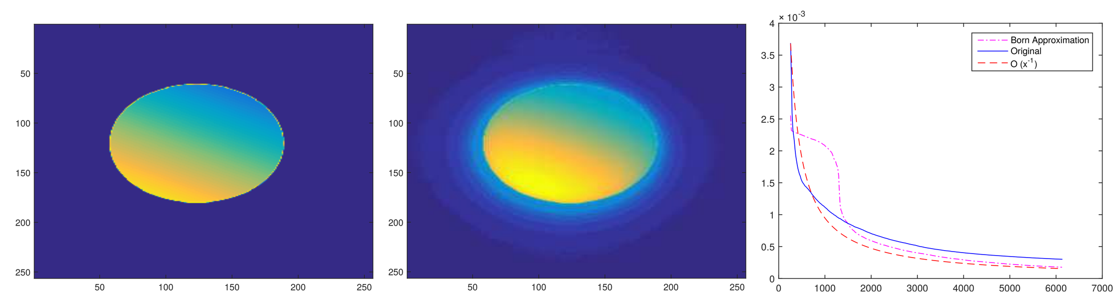


Figure: Decay of the shearlet coefficients (right) of the original function f (left) and its Born approximation f_B (middle). The decay is asymptotically of order $O(x^{-1})$.

Sparse Regularization

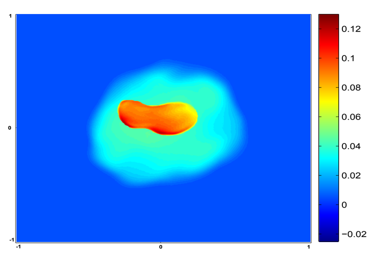
We employ the sparse approximation properties of shearlets to regularize the acoustic scattering problem.

- *Noisy measurements* $N^\varepsilon_{\text{meas}} \in HS(L^2(\Gamma_{\text{inc}}), L^2(\Gamma_{\text{meas}}))$, s.th. $\|N^\varepsilon_{\text{meas}} - N_{f^\dagger}\|_{\mathrm{HS}(L^2(\Gamma_{\text{inc}}),L^2(\Gamma_{\text{meas}}))} < \varepsilon$
- For a shearlet system $(\psi_{j,k,m})$, a *regularization parameter* $\alpha > 0$ and $1 \leq p \leq 2$, we minimize the following *Tikhonov functional*:

$$\mathscr{T}^\varepsilon_\alpha(f) := \|\mathscr{N}(f) - N^\varepsilon_{\text{meas}}\|^2_{\mathrm{HS}} + \alpha\|\langle f, \psi_{j,k,m}\rangle\|^p_{\ell^p}. \quad (1)$$

Numerical Examples

Task: Reconstruct a cartoon-like contrast function (right) by minimizing (1).



Comparison results

Compare with solutions of the L^1 Tikhonov functional

$$\mathscr{I}^\varepsilon_\alpha(f) := \|\mathscr{N}(f) - N^\varepsilon_{\text{meas}}\|^2_{\mathrm{HS}} + \alpha\|f\|_{L^1(B_R)}. \quad (2)$$

Experimental results

We solve (1) and (2) by *non-linear Landweber iteration.*

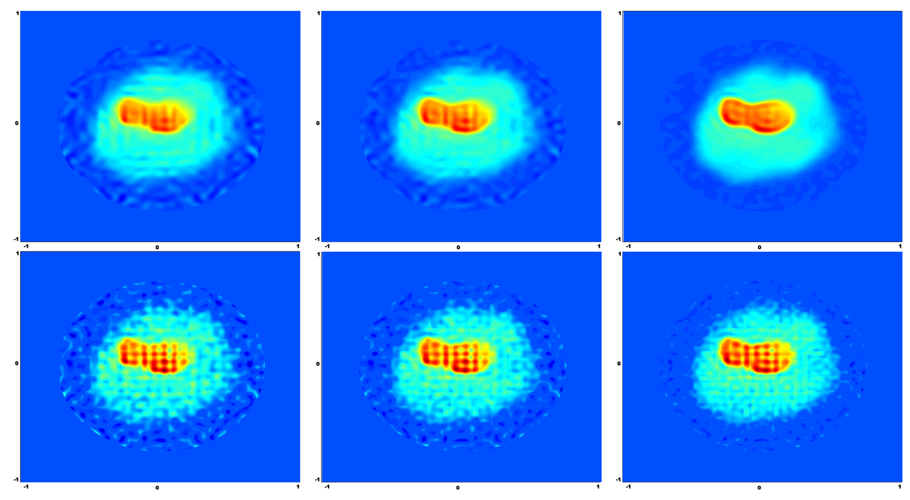


Figure: Reconstructed scatterers using shearlet regularization (top) and L^1 regularization (bottom). Noise levels $\varepsilon = 0.01, 0.005, 0.002$

$k_0 = 40$	Regularization method	Noise level	Rel. error	#iterations
1.	L^1 Tikhonov	0.02	0.1853	15
2.	Shearlets	0.02	**0.1450**	10
3.	No Penalty	0.02	0.2031	14
4.	L^1 Tikhonov	0.01	0.1204	40
5.	Shearlets	0.01	**0.0845**	24
6.	No Penalty	0.01	0.1471	47
7.	L^1 Tikhonov	0.005	0.1023	74
8.	Shearlets	0.005	**0.0603**	43
9.	No Penalty	0.005	0.1127	161
10.	L^1 Tikhonov	0.002	0.0797	303
11.	Shearlets	0.002	**0.0381**	100
12.	No Penalty	0.002	0.1014	978

Benefits of our approach

- Superior reconstructions to other penalties both *visually* and in terms of *relative errors*.
- *Less iterations* necessary to meet stopping criterion for the shearlet based regularization compared to other methods.

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- Education
  - Master thesis at TU Berlin
  - Research Assistant in DFG Research Center TRR 109
- Research interests
  - Applied harmonic analysis (Wavelets, Shearlets,..)
  - Inverse problems
  - Numerical analysis of PDEs
  - Signal and image processing