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Structure-Preserving Integration for LQR and Filtering

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Symplectic Integration

Hairer, Wanner, Lubich 2004 | Marsden, West 2001
Ober-Blöbaum, Junge, Marsden 2011

- Hamilton's equations for $H : T^*Q \to \mathbb{R}$ with external forces

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q} + f(q,p,u)$$

- flow of a (free) Hamiltonian system is *symplectic*
- **AIM**: preserve symplecticity in numerical simulations
- *symplectic integration scheme* defines a symplectic one-step map $\Phi_k : (q_k, p_k) \to (q_{k+1}, p_{k+1})$
- symplectic partitioned RK schemes, e.g. *Symplectic Euler*

$$\begin{pmatrix} q_{k+1} \\ p_{k+1} \end{pmatrix} = \begin{pmatrix} q_k \\ p_k \end{pmatrix} + h \cdot \begin{pmatrix} \frac{\partial}{\partial p} H(q_{k+1}, p_k) \\ -\frac{\partial}{\partial q} H(q_{k+1}, p_k) + f(q_{k+1}, p_k, u_k) \end{pmatrix}$$

Linear Quadratic Optimal Control

$$\min_u J(u) = \frac{1}{2}\int_0^T x^T Q x + u^T R u \, dt + \frac{1}{2} x(T)^T P_T x(T)$$

w.r.t. $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = x_0$.

- state-adjoint optimality system

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial \rho} = Ax + Bu, \quad x(0) = x^0, \quad u = -R^{-1}B^T\rho,$$
$$\dot{\rho} = -\frac{\partial \mathcal{H}}{\partial x} = -Qx - A^T\rho, \quad \rho(T) = P_T x(T),$$

for the *LQR problem's Hamiltonian*

$$\mathcal{H} = \frac{1}{2}(x^TQx + u^TRu) + \rho^T(Ax + Bu)$$

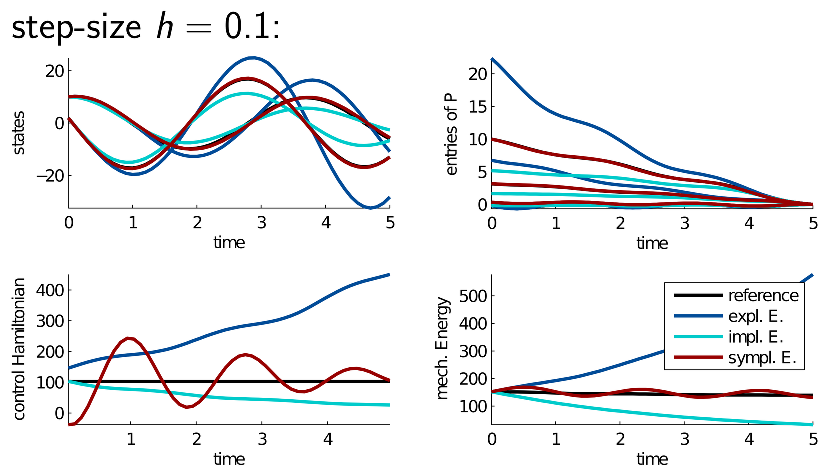
- Riccati eq. $\dot{P} = -PA - A^TP + PBR^{-1}B^TP - Q$, $P(T) = P_T$

(Stochastic) Harmonic Oscillator

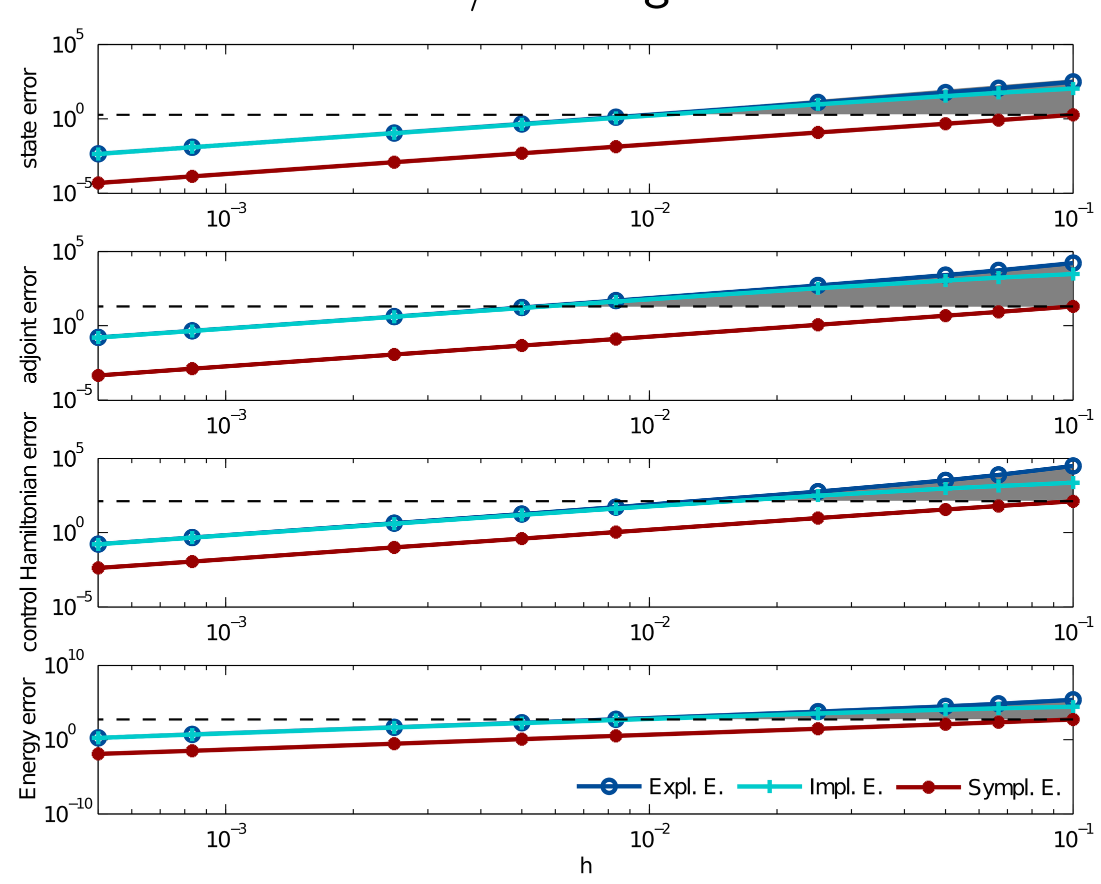
- comparison of 1st-order methods
- LQR problem:

$x^0 = (10, 2)$, $T = 5.0$,
$A = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,
$Q = \mathbb{I}$, $R = 100$, $P_T = 0$

step-size $h = 0.1$:

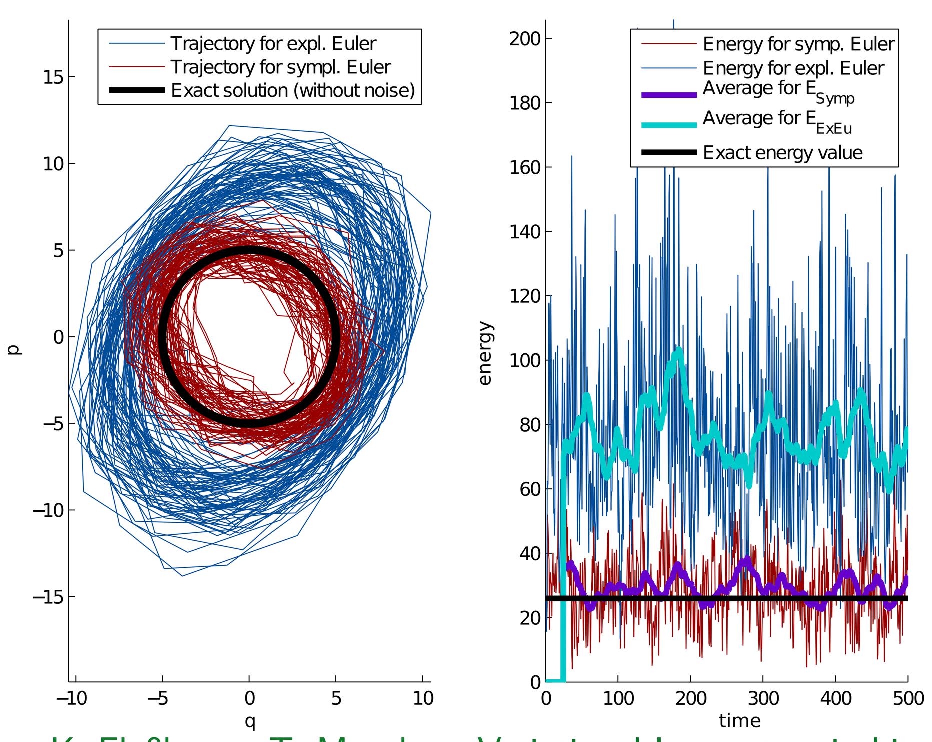


- symplectic discretization allows step-sizes one magnitude larger $\Rightarrow$ important in real-time robotic applications with low bandwidth actuation/ sensing



- Kalman filtering for stochastic harmonic oscillator

$\dot{x} = Ax + Gw$,
$z = Cx + v$, with
uncorrelated noise
$x(0) \sim \mathcal{N}(x^0, P_0)$,
$w \sim \mathcal{N}(0, Q)$, $v \sim \mathcal{N}(0, R)$
$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $G = \mathbb{I}$, $Q = \mathbb{I}$,
$R = \mathrm{diag}(1, 100)$, $C = \mathbb{I}$,
$x^0 = (5, 1)$, $T = 500$, $h = 0.5$



Symplectic Discrete-Time LQR

- **AIM:** preserve symplecticity of mechanics <u>and</u> state-adjoint system in discretization
- *aside:* standard discrete-time LQR corresponds to explicit Euler discretization of the dynamics and only preserves the symplectic state-adjoint system's flow
- transform LQR problem into Mayer form and apply e.g. symplectic Euler scheme to obtain optimization problem
- *The optimal solution satisfies a symplectic discretization of the Hamiltonian state-adjoint system*

$$\begin{pmatrix} q_{k+1} \\ p_{k+1} \\ \lambda_{k+1} \\ \mu_{k+1} \end{pmatrix} = \begin{pmatrix} q_k \\ p_k \\ \lambda_k \\ \mu_k \end{pmatrix} + h \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} q_{k+1} \\ p_k \\ \lambda_k \\ \mu_{k+1} \end{pmatrix},$$

with $(q_0, p_0) = (q^0, p^0)$ *and* $(\lambda_N, \mu_N)^T = P_T (q_N, p_N)^T$.

- *The discrete-time Riccati matrices are obtained iteratively from the scheme's one step map* Φ_k*, starting at* $P_N = P_T$*, by*

$$P_k = \left(P_{k+1}\Phi_k^{12} - \Phi_k^{22}\right)^{-1}\left(\Phi_k^{21} - P_{k+1}\Phi_k^{11}\right).$$

- sympl. state-adjoint scheme as in Ober-Blöbaum et al. 2011 | Hager 2000
- structure-preserving feedback controller can be used in projection-based nonlinear trajectory optimization
- applicable to linear (Kalman) filtering via duality principle

Kathrin Flaßkamp

- Education
  - Postdoc at Northwestern University, USA
  - Ph.D. and Diploma in math from U Paderborn
- Research interests
  - Optimal Control
  - Variational Integrators for Robotics
  - Hybrid Systems

References: K. Flaßkamp, T. Murphey: Variational Integrators in Linear Optimal Filtering, The 2015 American Control Conference, July 1–3, 2015, Chicago, IL
K. Flaßkamp, T. Murphey: Structure-Preserving Integration for Linear Optimal Control and Filtering of Mechanical Systems, submitted, 2015.