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Analysis of the Ensemble Kalman Filter for Inverse Problems

joint work with Andrew Stuart (University of Warwick)

Inverse Problem

Find the unknown data u in a separable Hilbert space X from noisy observations

$$y = \mathcal{G}(u) + \eta$$

- $X, Y, \mathcal{X}$ separable Hilbert spaces
- $G : X \mapsto \mathcal{X}$ the forward map
- $\mathcal{O} : \mathcal{X} \mapsto Y$ bounded, linear observation operator
- $\mathcal{G} : X \mapsto Y$ uncertainty-to-observation map, $\mathcal{G} = \mathcal{O} \circ G$
- $\eta \in Y$ the observational noise ($\eta \sim \mathcal{N}(0, \Gamma)$)
- μ_0 prior probability measure

The goal of computation is the posterior distribution

$$\mu(du) = \frac{1}{Z} \exp(\Phi(u))\mu_0(du)$$

with $\Phi : X \mapsto \mathbb{R}$, $\Phi(u) = \frac{1}{2}|y - \mathcal{G}(u)|^2_\Gamma$.

Ensemble Kalman Filter (EnKF)

- Fully Bayesian inversion is often too expensive.
- EnKF is widely used.
- Currently, very little analysis of the EnKF is available.

Aim: Build analysis of properties of EnKF

EnKF for Inverse Problem

Sequence of Interpolating Measures

For N, $h := 1/N$, we define a sequence of measures $\mu_n \ll \mu_0$, $n = 1, \dots, N$, which evolve the prior μ_0 into the posterior distribution $\mu_N = \mu$, by

$$\mu_{n+1}(du) = \frac{Z_n}{Z_{n+1}} \exp(-h\Phi(u))\mu_n(du) \Leftrightarrow \mu_{n+1} = L_n \mu_n$$

with operator L_n corresponding to application of Bayes' theorem and normalisation constant $Z_n = \int \exp(-nh\Phi(u))\mu_0(du)$.

Ensemble of Interacting Particles

Initial ensemble $\{u_0^{(j)}\}_{j=1}^J$ constructed by prior knowledge, $u^{(j)} \sim \mu_0$ iid for $J < \infty$.

Linearisation of $\mathbf{L_n}$ and approximation of $\mu_\mathbf{n}$ by a J-particle Dirac measure leads to the EnKF method.

Update of the EnKF for Inverse Problems

$$u_{n+1}^{(j)} = u_n^{(j)} + C_{n+1}^{up}(C_{n+1}^{pp} + \frac{1}{h}\Gamma)^{-1}(y_{n+1}^{(j)} - \mathcal{G}(u_n^{(j)}))$$

with empirical covariances $C_{n+1}^{up} = \frac{1}{J}\sum_{j=1}^J u_n^{(j)} \otimes \mathcal{G}(u_n^{(j)}) - \overline{u}_n \otimes \overline{\mathcal{G}}(u_n)$, $C_{n+1}^{pp} = \frac{1}{J}\sum_{j=1}^J \mathcal{G}(u_n^{(j)}) \otimes \mathcal{G}(u_n^{(j)}) - \overline{\mathcal{G}}(u_n) \otimes \overline{\mathcal{G}}(u_n)$, mean $\overline{u}_n = \frac{1}{J}\sum_{j=1}^J u_n^{(j)}$, $\overline{\mathcal{G}}(u_n) = \frac{1}{J}\sum_{j=1}^J \mathcal{G}(u_n^{(j)})$ and observations $y_{n+1}^{(j)} = y + \eta_{n+1}^{(j)}$, $\eta_{n+1}^{(j)} \sim N(0, \frac{1}{h}\Gamma)$.

Continuous Time Limit

Limiting SDE

Interpreting the iterate as $u_n^{(j)} = u^{(j)}(nh)$ gives

$$du^{(j)} = C^{up}\Gamma_0^{-1}(y - \mathcal{G}(u^{(j)}))\,dt + C^{up}\Gamma_0^{-\frac{1}{2}}\,dW^{(j)},$$

where $W^{(1)}, \dots, W^{(J)}$ are pairwise independent cylindrical Wiener processes.

- Deriving the continuous time limit allows to determine the asymptotic behaviour of important quantities of the algorithm.
- The continuous approach offers the possibility to improve the performance of the method by choosing appropriate numerical discretisation schemes based on the properties of the solution.
- Based on the analysis of the SDE, strategies to adaptively control the approximation quality of the subspace spanned by the ensemble can be developed.
- This approach is not limited to the linear case and may give also some insights for the nonlinear case.

Groundwater Flow Problem

Enabling Quantification of
EQUIP
Uncertainty for Inverse Problems

Model Problem

We consider steady groundwater flow in a two-dimensional confined aquifer governed by the following equation

$$-\nabla \cdot \kappa \nabla h = f$$

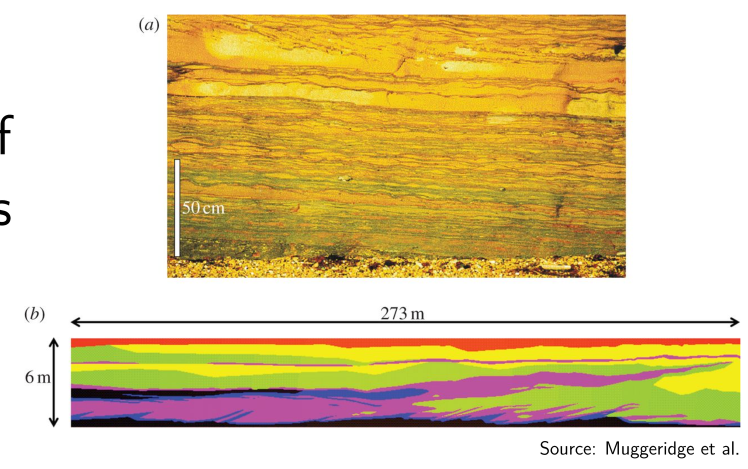
with piezometric head h, source f and hydraulic conductivity κ.

Uncertainty in the hydraulic conductivity κ

Typical models for κ are log-normal priors or more realistic multipoint prior, which are able to capture the channelized nature of rock formations in the subsurface.

Measurements

Measurements are comprised of $G_j(\kappa) = h(x_j)$ for some set of points $\{x_j\}_{j=1}^K$ in the physical domain.



Source: Muggeridge et al.

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- Education
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- Research Interests
  - Numerical Analysis, UQ, Inverse Problems, Robust Optimization

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